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# Approximation with Reciprocals of Polynomials on Compact Sets*†

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## 1. INTRODUCTION

Let $X$ be a compact subset of the real line and denote by $C(X)$ the class of all real valued continuous functions defined on $X$. Norm $C(X)$ with the uniform norm, i.e., for all $f \in C(X)$, $\|f\| = \max\{|f(x)| : x \in X\}$. Let $n$ be a positive integer and set $R_n^0(X) = \{1/p \colon p \in \Pi_n, p(x) > 0, \forall x \in X\}$ where $\Pi_n$ denotes the set of all real algebraic polynomials of degree $\leqslant n$. Note that $R_n^0(X)$ consists of the positive elements of the set usually denoted by $R_n^0(X)$.

In this paper, we shall study the problem of approximation of positive functions in $C(X)$ by elements of $R_n^0(X)$. The emphasis of this study is as follows. First, we wish to contrast this setting with that of approximation by elements of $R_n^m(X)$, $m \geqslant 1$, $n \geqslant 1$, in $C(X)$. Basically, there is one major difference, namely, that existence holds for this case; whereas, this is not true for $R_n^m(X)$, $m \geqslant 1$, $n \geqslant 1$ and $X$ not an interval. In addition, when $X$ is not an interval the proof of existence is very long and tedious. Next, we shall observe that the usual characterization (alternation) and uniqueness results hold for this problem using the standard arguments. Finally, we shall discuss the computation of best approximations from $R_n^0(X)$.

These results will be used in a forthcoming paper on uniform approximation on $[0, \infty)$ with reciprocals of polynomials. See [3, 4, 5, 9, 15, 19] for

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various recent studies concerning certain aspects of this problem. Since we are interested in applying these results to this setting, we shall approximate $1/f$ in what follows.

## 2. EXISTENCE

As noted in the introduction, it is well known that best rational approximants from $R_n^m(X)$, $m \geq 1$, $n \geq 1$, and $X$ not an interval need not exist. Indeed, let $X$ be a finite subset of $[0, 1]$, such that $0 \in X$ and $\text{card}(X) \geq n + m + 2$. Define $f \in C(X) \sim R_n^m(X)$ by $f(0) = 1$ and $f(x) = 2$ for $x \in X \sim \{0\}$. Let $r_k(x)$ be defined by $r_k(x) = (2kx + 1)/(kx + 1)$. Then $\lim_{k\to\infty} \| f - r_k \| = 0$ showing that no best approximation exists for $f$ from $R_n^m(X)$.

In what follows, let $n$ be a nonnegative integer and let $X$ be a compact subset of the real line with $\text{card}(X) \geq n + 2$. Set $K = \{p \in \Pi_n : p(x) > 0, \forall x \in X\}$, let $f \in C(X)$, $f \notin K$ with $f(x) > 0$ for all $x \in X$ and set $\Delta = \inf\{\|(1/f) - (1/p)\| : p \in K\}$. We have the following lemma.

LEMMA 1. *Let $X, f, \Delta$ be as defined above. Then $\Delta > 0$.*

*Proof.* Suppose $\Delta = 0$. Then, setting $(1/p) = (c/q)$, $c$ a constant, $q \in K$, $\| q \| = 1$, there is a sequence of polynomials $\{q_n\} \subset K$ such that $1 \geq \|(1/f) - (c_n/q_n)\| \to 0$. Since $| c_n | \leq 1 + (1/m)$, $m = \min\{| f(x)| : x \in X\}$, we may assume that $c_n \to c^* \geq 0$ and $q_n \to q^*$ uniformly, with $q^* \in \Pi_n$ and $\| q^* \| = 1$. We claim that $c^* > 0$. Suppose $c^* = 0$, and let $x \in X$ be such that $q^*(x) > 0$ (such an $x$ exists since $q^* \not\equiv 0$ and $\text{card}(X) \geq n + 2$). Then we have

$$0 < \frac{1}{f(x)} = \frac{1}{f(x)} - \frac{0}{q^*(x)} = \lim_{n\to\infty} \left(\frac{1}{f(x)} - \frac{c_n}{q_n(x)}\right) \leq \lim_{n\to\infty} \left\| \frac{1}{f} - \frac{c_n}{q_n} \right\| = 0$$

which is a contradiction. Hence $c^* > 0$. Since $\| c_n/q_n \| \leq 1 + \| 1/f \|$ we have that $q^*(x) > 0$ for all $x \in X$. Thus $c_n/q_n \to c^*/q^* \in K$ uniformly and

$$\left\| \frac{1}{f} - \frac{c^*}{q^*} \right\| = \lim_{n\to\infty} \left\| \frac{1}{f} - \frac{c_n}{q_n} \right\| = 0,$$

hence $f = q^*/c^* \in K$, which is a contradiction. Therefore $\Delta > 0$.

THEOREM 1. *Let $n$ be a nonnegative integer and let $X$, $K$, $f$, and $\Delta$ be defined as above. Then there exists a $p^* \in K$ such that $\|(1/f) - (1/p^*)\| = \Delta$.*

*Proof.* Since $X$ is compact, $f > 0$ on $X$, there exist positive constants $m$, $M$ such that $m \leq 1/f(x) \leq M$ for all $x \in X$. For $p \in K$ we write $1/p = c/q$ where $q(x) > 0$ for all $x \in X$, $c > 0$, and $\| q \| = 1$. Then, as in Lemma 1, there exist sequences $\{c_n\}$, $\{q_n\}$ such that $c_n > 0$ for all $n$, $q_n(x) > 0$ for all $x \in X$ with $\| q_n \| = 1$ satisfying

(i) $1 + \Delta \geqslant \|(1/f) - (c_n/q_n)\|$,

(ii) $\lim_{n\to\infty} \|(1/f) - (c_n/q_n)\| = \Delta$,

(iii) $| c_n | \leqslant 1 + \Delta + M$.

By extracting subsequences $\{c_{n_\nu}\}$ and $\{q_{n_\nu}\}$ of $\{c_n\}$ and $\{q_n\}$, respectively, there exists $c^* \geqslant 0$ and $q^* \in \Pi_n$ with $q^*(x) \geqslant 0$ for all $x \in X$ and $\| q^* \| = 1$ such that $c_{n_\nu} \to c^*$ and $q_{n_\nu}(x) \to q^*(x)$ uniformly in $X$.

We now claim that $c^* > 0$. Indeed, suppose $c^* = 0$ and let $Z = \{x_1, ..., x_k\} \subset X$, $k \leqslant n$, be all the zeros of $q^*$ contained in $X$. Now, if $Z = \varnothing$, then $q^*(x) > 0$ for all $x \in X$ holds and we have for each $x \in X$

$$\frac{1}{f(x)} = \frac{1}{f(x)} - \frac{0}{q^*(x)} = \lim_{\nu\to\infty} \left(\frac{1}{f(x)} - \frac{c_{n_\nu}}{q_{n_\nu}(x)}\right) \leqslant \lim_{\nu\to\infty} \left\|\frac{1}{f} - \frac{c_{n_\nu}}{q_{n_\nu}}\right\| = \Delta. \quad (1)$$

Thus, taking $\tilde{p} = 2/\Delta$, the inequality

$$\left|\frac{1}{f(x)} - \frac{1}{\tilde{p}(x)}\right| = \left|\frac{1}{f(x)} - \frac{\Delta}{2}\right| \leqslant \frac{\Delta}{2}$$

holds for each $x \in X$. Since $\tilde{p} \in K$, this contradicts our assumption that $\inf_{p\in\Pi_n}\{\|(1/f) - (1/p)\| : p \in K\} = \Delta > 0$. Therefore, we assume that $Z \neq \varnothing$. Partition $Z$ into two subsets $Z = I \cup J$ where $I = \{x \in Z : x \text{ is an isolated point of } X\}$, and $J = Z \sim I$. Now, for $x \in X \sim Z$ we have that $q^*(x) > 0$ so $1/f(x) \leqslant \Delta$ holds by (1). Also, since $f \in C(X)$ and $y \in J$ implies that $y$ is a limit point of $X \sim Z$ we have, by continuity, that $1/f(y) \leqslant \Delta$ holds. Now, if $I = \varnothing$, then our argument for the case $Z = \varnothing$ yields our desired contradiction. Hence, let us assume that $I = \{y_1 < y_2 < \cdots < y_\mu\}$, $\mu \geqslant 1$. Now, let $a < b$ be such that $X \subset (a, b)$. Construct open intervals $(\alpha_\nu, \beta_\nu)$ as follows (observing that $X \sim I$ is a compact subset of $X$): set $\alpha_1 = \max\{a, \max\{x: x \in X \sim I \text{ and } x < y_1\}\}$ and $\beta_1 = \min\{b, \min\{x: x \in X \sim I \text{ and } x > y_1\}\}$. If $\beta_1 > y_\mu$ stop this process with the interval $(\alpha_1, \beta_1)$. If $\beta_1 < y_\mu$, then there exists an integer $i_1$, $1 \leqslant i_1 \leqslant \mu - 1$ such that $y_{i_1} < \beta_1 < y_{i_1+1}$. In this case, set $\alpha_2 = \max\{x: x \in X \sim I \text{ and } x < y_{i_1+1}\}$ and $\beta_2 = \min\{b, \min\{x: x \in X \sim I \text{ and } x > y_{i_1+1}\}\}$. Note that we must have $\alpha_2 \geqslant \beta_1$ as $\beta_1 \in X \sim I$ and $\beta_1 < y_{i_1+1}$. Once again, if $\beta_2 > y_\mu$, stop this process. If $\beta_2 < y_\mu$, we continue and, since $\mu$ is finite, this construction must end after, say $\nu \leqslant \mu$ steps, giving $\nu$ pairwise disjoint open intervals $(\alpha_1, \beta_1), ..., (\alpha_\nu, \beta_\nu)$ where $a \leqslant \alpha_1 < \beta_1 \leqslant \alpha_2 < \beta_2 \leqslant \cdots \leqslant \alpha_\nu < \beta_\nu \leqslant b$, $(\alpha_i, \beta_i) \cap I \neq \varnothing$ and $(\alpha_i, \beta_i) \cap (X \sim I) = \varnothing$ for $i = 1, ..., \nu$. For convenience, let us assume that $(\alpha_r, \beta_r) \cap I = \{y_{i_{r-1}+1}, ..., y_{i_r}\}$ for $r = 1, ..., \nu$ with $i_0 = 0$. Now we will construct a new set of points $I'$

where $I' \cap X = \varnothing$ as follows. If $(\alpha_r, \beta_r)$ is such that $a < \alpha_r, \beta_r < b$ and $(\alpha_r, \beta_r) \cap I = \{y_{i_{r-1}+1}, \ldots, y_{i_r}\}$ consists of an odd number of points, then set $y'_{i_r} = \frac{1}{2}(\beta_r + y_{i_r})$ and require that $y'_{i_r} \in I'$. Therefore, $I'$ can consist of at most $\nu$ points. For convenience, set $I' = \{y_1', \ldots, y_\gamma'\}$, $\gamma \leqslant \nu$, if $I' \neq \varnothing$. Also, in this case, note that $q^*(x)$ must vanish at $y_{i_{r-1}+1}, \ldots, y_{i_r}$ and $q^*(\alpha_r) > 0$, $q^*(\beta_r) > 0$ both hold implying that $q^*$ must have either one $y_i$, $i_{r-1} + 1 \leqslant i \leqslant i_r$ as a zero of even order (at least two) or have at least one more simple zero in $(\alpha_r, \beta_r)$. That is, $q^*$ must have at least $i_r - i_{r-1} + 1$ zeros in $(\alpha_r, \beta_r)$. Set

$$
\begin{aligned}
v(x) &= \prod_{i=1}^{\gamma} (y_i' - x), \qquad I' \neq \varnothing \\
&= 1, \qquad\qquad I' = \varnothing .
\end{aligned} \tag{2}
$$

Next, we shall construct a sequence of polunomials $\{\varphi_i^N\}_{N=1}^{\infty}$ corresponding to each interval $(\alpha_i, \beta_i)$, $i = 1, \ldots, \nu$. First, set

$$
\omega_j = \frac{f(y_j)}{N \prod_{i=1}^{j-1} (y_j - y_i) \prod_{i=j+1}^{\mu} (y_i - y_j) \mid v(y_j)} \tag{3}
$$

for $j = 1, \ldots, \mu$ where $\prod_1^0 = \prod_{\mu+1}^{\mu} = 1$ and note that $\omega_j \to 0$ as $N \to \infty$. Let us first consider the interval $(\alpha_1, \beta_1)$ where $(\alpha_1, \beta_1) \cap I = \{y_1, \ldots, y_{i_1}\}$, $1 \leqslant i_1 \leqslant \mu$. The precise form of $\varphi_1^N$ will depend upon the structure of $(\alpha_1, \beta_1)$ although, in all cases, the polynomials $\varphi_1^N$ will have certain essential properties. Therefore, we must consider cases.

*Case* 1. $\alpha_1 = a$, $i_1 = 2m + 1$, $m \geqslant 0$. Note that in this case we must have $\beta_1 < b$ since $\operatorname{card}(X) \geqslant n + 2$. Furthermore, this interval gives no contribution to $I'$. Set

$$
\varphi_1^N(x) = -\prod_{j=0}^{m} [(y_{2j+1} - \omega_{2j+1}) - x] \prod_{j=1}^{m} [(y_{2j} + \omega_{2j}) - x]
$$

where $\omega_j$ is defined by (3). Now, since $\omega_j \to 0$ as $N \to \infty$ for all $j$ and $a = \alpha_1 < y_1 < \cdots < y_{i_1} < \beta_1$, we can select an $N_1$ such that $N \geqslant N_1$ implies that

$$
\begin{aligned}
a &< y_1 - \omega_1 < y_1 < y_2 < y_2 + \omega_2 < \cdots \\
&< y_{i_1-1} < y_{i_1-1} + \omega_{i_1-1} < y_{i_1} - \omega_{i_1} < y_{i_1} .
\end{aligned}
$$

Thus, for $N \geqslant N_1$, we have that $\varphi_1^N(x)$ is positive at $y_1, \ldots, y_{i_1}$, and $\varphi_1^N(x) > 0$ for $x \geqslant \beta_1$ (which implies that $\varphi_1^N(x) > 0$ for all $x \in X$). Let $\rho(s)$ denote the number of zeros of $q^*(x)$ in the interval $(\alpha_s, \beta_s)$, $s = 1, \ldots, \nu$ where we count a zero of order $p$ as $p$ zeros. Then we have $\partial\varphi_1^N \leqslant \rho(1)$ in $(\alpha_1, \beta_1)$. Also, setting $\epsilon_i = y_i - \omega_i$ for $i$ odd and $\epsilon_i = y_i + \omega_i$ for $i$ even and noting that $\epsilon_i \to y_i$ as $N \to \infty$, we have, for $1 \leqslant i \leqslant i_1$, that

$$1/N\varphi_1^N(y_i) = \left[N(-1)^{i+1}\,\omega_i \prod_{j=1}^{i-1} (\epsilon_j - y_i) \prod_{j=i+1}^{i_1} (\epsilon_j - y_i)\right]^{-1}$$

$$= \frac{\prod_{j=1}^{i-1} (y_i - y_j) \prod_{j=i+1}^{\mu} (y_j - y_i)\,|\,v(y_i)|}{f(y_i) \prod_{j=1}^{i-1} (y_i - \epsilon_j) \prod_{j=i+1}^{i_1} (\epsilon_j - y_i)}.$$

Now, using (2) and (3), we have

$$1/N\varphi_1^N(y_i) \to f(y_i)^{-1} \prod_{j=i_1+1}^{\mu} (y_j - y_i) \prod_{j=1}^{\nu} (y_j' - y_i)$$

$$\text{as} \quad N \to \infty, \quad 1 \leqslant i \leqslant i_1 .$$

Also, for $x \in X$ satisfying $x > y_{i_1}$, we have (since $i_1$ is assumed odd) that $\varphi_1^N(x) = -\prod_{j=1}^{i_1} (\epsilon_j - x) = \prod_{j=1}^{i_1} (x - \epsilon_j) \to \prod_{j=1}^{i_1} (x - y_j)$ as $N \to \infty$.

*Case* 2. $\alpha_1 = a, i_1 = 2m, m \geqslant 1$. Once again we have $\beta_1 < b$ and $(\alpha_1, \beta_1) \cap I' = \varnothing$. Set $\varphi_1^N(x) = \prod_{j=0}^{m-1} [(y_{2j+1} + \omega_{2j+1}) - x] \prod_{j=1}^{m} [(y_{2j} - \omega_{2j}) - x]$. Again it follows that we can select an $N_1$ such that $N \geqslant N_1$ implies that $\varphi_1^N(y_i) > 0, 1 \leqslant i \leqslant i_1$, $\varphi_1^N(x) > 0$ for $x \geqslant \beta_1$ (which implies that $\varphi_1^N(x) > 0$ for all $x \in X$) and $\partial\varphi_1^N \leqslant \rho(1)$. Setting $\eta_i = y_i + \omega_i$ for $i$ odd and $\eta_i = y_i - \omega_i$ for $i$ even, we have for $y_i$, $1 \leqslant i \leqslant i_1$, that $1/N\varphi_1^N(y_i) = 1/(N(-1)^{i-1} \omega_i \prod_{j=1 j\neq i}^{i_1} (\eta_j - y_i))$. Again, using (2) and (3) we have (since $i_1$ is even) that $1/N\varphi_1^N(y_i) \to f(y_i)^{-1} \prod_{j=i_1+1}^{\mu} (y_j - y_i) \prod_{j=1}^{\nu} (y_j' - y_i)$ as $N \to \infty$. Also, for $x \in X$ with $x > y_{i_1}$, we have that $\varphi_1^N(x) = \prod_{j=1}^{i_1} (\eta_j - x) = \prod_{j=1}^{i_1} (x - \eta_j) \to \prod_{j=1}^{i_1} (x - y_j)$ as $N \to \infty$. Note that $\varphi_1^N$ has the same limit, as $N \to \infty$, for each of these two cases.

Next we consider the case where $\alpha_1 > a$. In this case, the contribution of $\varphi_1^N$ is identical with that of $\varphi_r^N$ for $(\alpha_r, \beta_r)$, $r = 2,..., \nu$. Thus, we consider the construction of $\varphi_r^N$ for $(\alpha_r, \beta_r)$ where $r = 1,..., \nu$. Here we must consider an additional four cases. For convenience set $l = i_{r-1}$ and $k = i_r - i_{r-1}$ so that $(\alpha_r, \beta_r) \cap I = \{y_{i_{r-1}+1}, ..., y_{i_r}\} = \{y_{l+1}, ..., y_{l+k}\}$.

*Case* 3. $\alpha_r > a$, $\beta_r < b$, $r = 1,..., \nu$; $k = 2m + 1$, $m \geqslant 0$. Note that in this case $y_{i_r}' \equiv y_{l+k}' \in I'$ and $\rho(r) \geqslant 2m + 2$. Set

$$\varphi_r^N(x) = (y_{l+k}' - x) \prod_{j=0}^{m} [(y_{l+2j+1} + \omega_{l+2j+1}) - x] \prod_{j=1}^{m} [(y_{l+2j} - \omega_{l+2j}) - x]. \tag{4}$$

Once again, there exists an $N_r$ such that for $N \geqslant N_r$ we have

(i) $\varphi_r{}^N(x) > 0,\ x \leqslant \alpha_r$,

(ii) $\varphi_r{}^N(y_i) > 0,\ i = l+1,\ldots, l+k$,

(iii) $\varphi_r{}^N(x) > 0,\ x \geqslant \beta_r$.

(Note that (iii) follows from the even number of linear factors in (4), all of which are negative.) Therefore, $\varphi_r{}^N(x) > 0$ for all $x \in X$. Setting $\rho_i = y_{l+i} + \omega_{l+i}$ for $i$ odd and $\rho_i = y_{l+i} - \omega_{l+i}$ for $i$ even, we have for $x \leqslant \alpha_r$

$$\varphi_r{}^N(x) = \prod_{i=1}^{k} (\rho_i - x)(y'_{l+k} - x)$$

$$\to \prod_{j=i_{r-1}+1}^{i_r} (y_j - x)(y'_{i_r} - x) \qquad \text{as} \quad N \to \infty.$$

Similarly for $x \geqslant \beta_r$ (since $k$ is odd)

$$\varphi_r{}^N(x) = \prod_{i=1}^{k} (\rho_i - x)(y'_{l+k} - x)$$

$$\to \prod_{j=i_{r-1}+1}^{i_r} (x - y_j)(x - y'_{i_r}) \qquad \text{as} \quad N \to \infty.$$

Finally, for $y_i$, $i_{r-1} + 1 \leqslant i \leqslant i_r$, we have, using (2) and (3) (assuming $y'_{i_r} = y_\theta'$ for some $\theta$, $1 \leqslant \theta \leqslant r$),

$1/N\varphi_r{}^N(y_i)$

$$= \left[N(-1)^{i-i_{r-1}-1}\,\omega_i \prod_{j=1}^{i-i_{r-1}-1} (\rho_j - y_i) \prod_{j=i-i_{r-1}+1}^{k} (\rho_j - y_i)(y'_{i_r} - y_i)\right]^{-1}$$

$$= \frac{\prod_{j=1}^{i-1} (y_i - y_j) \prod_{j=i+1}^{\mu} (y_j - y_i) \mid \prod_{j=1}^{\nu} (y_j' - y_i)\mid}{f(y_i) \prod_{j=1}^{i-i_r-1} (y_i - \rho_j) \prod_{j=i-i_{r-1}+1}^{k} (\rho_j - y_i)(y_\theta' - y_i)}$$

$$\to f(y_i)^{-1} \prod_{j=1}^{i_r-1} (y_i - y_j) \prod_{j=i_r+1}^{\mu} (y_j - y_i) \prod_{j=1}^{\theta-1} (y_\theta' - y_j) \prod_{j=\theta+1}^{\nu} (y_j' - y_\theta')$$

as $N \to \infty$.

*Case* 4. $\alpha_r > a$, $\beta_r < b$, $k = 2m$, $m \geqslant 1$. Note that in this case $(\alpha_r, \beta_r)$ gives no contribution to $I'$. Set

$$\varphi_r{}^N(x) = \prod_{j=0}^{m-1} [(y_{l+2j+1} + \omega_{l+2j+1}) - x] \prod_{j=1}^{m} [(y_{l+2j} - \omega_{l+2j}) - x].$$

As before, there exists an $N_r$ such that $N \geqslant N_r$ implies $\varphi_r{}^N(x) > 0$ for all $x \in X$. Using $\rho_i$ as defined in Case 3, we have, for $x \leqslant \alpha_r$ $\varphi_r{}^N(x) = \prod_{j=1}^k (\rho_j - x) \to \prod_{j=i_{r-1}+1}^{i_r} (y_j - x)$ as $N \to \infty$, and for $x \geqslant \beta_r$ $\varphi_r{}^N(x) = \prod_{j=1}^k (\rho_j - x) \to \prod_{j=i_{r-1}+1}^{i_r} (x - y_j)$ as $N \to \infty$. Finally, for $y_i$, $i_{r-1} + 1 \leqslant i \leqslant i_r$, we have, using (2) and (3).

$$1/N\varphi_r{}^N(y_i) = \left[N(-1)^{i-i_{r-1}+1} \omega_i \prod_{j=1}^{i-i_{r-1}-1} (\rho_j - y_i) \prod_{j=i-i_{r-1}+1}^{k} (\rho_j - y_i)\right]^{-1}$$

$$= \frac{\prod_{j=1}^{i-1} (y_i - y_j) \prod_{j=i+1}^{\mu} (y_j - y_i) \prod_{j=1}^{\gamma} | y_j' - y_i |}{f(y_i) \prod_{j=1}^{i-i_{r-1}-1} (y_i - \rho_j) \prod_{j=i-i_{r-1}-1}^{k} (\rho_j - y_i)}$$

$$\to f(y_i)^{-1} \prod_{j=1}^{i_{r-1}} (y_i - y_j) \prod_{j=i_r+1}^{\mu} (y_j - y_i) \prod_{j=1}^{\gamma} | y_j' - y_i |$$

$$\text{as} \quad N \to \infty.$$

*Case* 5. $\alpha_r > a$, $\beta_r = b$, $i_r - i_{r-1} = k = 2m + 1$, $m \geqslant 0$. Note that in this case $r = \nu$, $i_\nu = \mu$, and $(\alpha_\nu, \beta_\nu)$ gives no contribution to $I'$. Set $\varphi_\nu{}^N(x) = \prod_{j=0}^m [(y_{l+2j+1} + \omega_{l+2j+1}) - x] \prod_{j=1}^m [(y_{l+2j} - \omega_{l+2j}) - x]$ where $\omega_i$ is defined by (3). As before, there exists an $N_r$ such that $N \geqslant N_r$ implies $\varphi_\nu{}^N(x) > 0$ for all $x \in X$. Using $\rho_i$ as defined in Case 3, we have for $x \leqslant \alpha_\nu$, $\varphi_\nu{}^N(x) = \prod_{i=1}^k (\rho_i - x) \to \prod_{i=i_{\nu-1}+1}^{\mu} (y_i - x)$ as $N \to \infty$. Also, for $y_i$, $i_{r-1} + 1 \leqslant i \leqslant \mu$

$$\frac{1}{N\varphi_\nu{}^N(y_i)} = \left[N(-1)^{i-i_{\nu-1}+1} \omega_i \prod_{\substack{j=1 \\ j \neq i-i_{r-1}}}^{k} (\rho_j - y_i)\right]^{-1}$$

$$\to f(y_i)^{-1} \prod_{j=1}^{i_{\nu-1}} (y_i - y_j) \prod_{j=1}^{\gamma} | y_j' - y_i |$$

as $N \to \infty$ using the same argument as before.

*Case* 6. $\alpha_r > a$, $\beta_r = b$ $(r = \nu$, $i_r = \mu)$, $\mu - i_{r-1} = k = 2m$, $m \geqslant 1$. Again, the interval $(\alpha_\nu, \beta_\nu)$ gives no contribution to $I'$. Set $\varphi_\nu{}^N(x) = \prod_{j=0}^{m-1} [(y_{l+2j+1} + \omega_{l+2j+1}) - x] \prod_{j=1}^m [(y_{l+2j} - \omega_{l+2j}) - x]$. As before, we have $\varphi_\nu{}^N(x) > 0$ for all $x \in X$. $\varphi_\nu{}^N(x) \to \prod_{j=i_{\nu-1}+1}^k (y_j - x)$ as $N \to \infty$, for $x \leqslant \alpha_\nu$. For $y_i$, $i_{\nu-1} + 1 \leqslant i \leqslant \mu$, $1/N\varphi_\nu{}^N(y_i) \to f(y_i)^{-1} \prod_{j=1}^{i_{\nu-1}} (y_i - y_j) \prod_{j=1}^{\gamma} | y_j' - y_i |$. Recalling that $\rho(s)$ is the number of zeros of $q^*(x)$ in the interval $(\alpha_s, \beta_s)$, $s = 1, \ldots, \nu$, we observe that in Cases 4, 5, and 6, $\partial\varphi_r{}^N(x) \leqslant \rho(r)$. Therefore, we set $p_N(x) = \prod_{s=1}^{\nu} \varphi_s{}^N(x)$, where $\varphi_s{}^N(x)$ is constructed with respect to the interval $(\alpha_s, \beta_s)$, $s = 1, \ldots, \nu$ as described above (depending upon $(\alpha_s, \beta_s)$). Since $\partial\varphi_s{}^N(s) \leqslant \rho(s)$, these intervals (by construction) are pairwise disjoint and $\| q^* \|_X = 1$, we have that $\partial p_N \leqslant \partial q^*$. Also, there exists an $N$ such that $N \geqslant N^*$ implies that $\varphi_s^N(x) > 0$ for all $x \in X$, $s = 1,\ldots, \nu$ so that $p_N \in K$ for $N \geqslant N^*$. Furthermore, for $x \in X \sim I$,

$$\lim_{N\to\infty} \frac{1}{Np_N(x)} = \left(\lim_{N\to\infty} \frac{1}{N}\right) \prod_{s=1}^{\nu} \lim_{N\to\infty} \frac{1}{\varphi_s^N(x)}$$

$$= \left(\lim_{N\to\infty} \frac{1}{N}\right)\left(\prod_{i=1}^{\mu} | y_i - x | \prod_{i=1}^{\gamma} | y_i' - x |\right)^{-1} = 0$$

since the quantity in parentheses on the right is not equal to zero. Also, since $\mathrm{dist}(I \cup I', X \sim I) = c > 0$, we have that the above convergence is uniform in $X \sim I$. Finally, for $y_i \in I$ (say $y_i \in (\alpha_r, \beta_r)$) we have that

$$\lim_{N\to\infty} \frac{1}{Np_N(y_i)} = \left(\lim_{N\to\infty} \prod_{\substack{s=1\\ s\neq r}}^{\nu} \frac{1}{\varphi_s^N(y_i)}\right)\left(\lim_{N\to\infty} \frac{1}{N\varphi_r^N(y_i)}\right) = f(y_i)^{-1}.$$

Thus, by selecting $N \geqslant N^*$ sufficiently large so that $|[1/f(y_i)] - [1/Np_N(y_i)]| < \Delta$, $i = 1,\ldots, \mu$ and $| 1/Np_N(x)| < \Delta$ holds for all $x \in X \sim I$ we have that $\|(1/f) - (1/Np_N)\|_X < \Delta$ (recall that $| 1/f(x)| \leqslant \Delta$ on $X \sim I$), which is our desired contradiction. Hence, we must have $c^* > 0$. Thus, $q^*(x) > 0$ for all $x \in X$. Furthermore, $(c_\nu/q_\nu) \to (c^*/q^*)$ uniformly in $X$ as $\nu \to \infty$ so that $\|(1/f) - (c^*/q^*)\| = \lim_{\nu\to\infty} \|(1/f) - (c_\nu/q_\nu)\| = \Delta$. This, in turn implies that $p^* = q^*/c^*$ is our desired best approximation from $K$.

Finally, we would like to close this section with results on characterization and uniqueness. Using the standard argument [1] for alternation of best rational approximants on an interval we have:

THEOREM 2. *Let $f \in C(X)$, where $X$ is a compact subset of the real line. Let $n \geqslant 0$ be a given integer and set $R_n^0(X) = \{1/p(x): p \in \Pi_n, p(x) > 0$, for all $x \in X\}$. Assume $1/f \notin R_n^0(X)$. Then a necessary and sufficient condition that $1/p^*$ is a best approximation to $1/f$ on $X$ from $R_n^0(X)$ is that the error curve $e^*(1/f) = 1/f - 1/p^*$ alternate at least $n + 1$ times.*

It should be remarked that this theorem is also valid for $R_n^m(X) = \{r = p/q : p \in \Pi_m, q \in \Pi_n, q > 0$ on $X$, and $(p, q) = 1\}$ where $(p, q)$ denotes the greatest common polynomial divisor of $p$ and $q$ with $n + m + 2 - d$ alternating extreme points needed, $d = \min(m - \partial p, n - \partial q)$. The proof of sufficiency follows as in the $R_n^m[a, b]$ case [1]. The arguments of necessity also apply here; however, a certain amount of care must be taken. Namely, if one assumes that $r^* = p^*/q^*$ is a best approximation having less than the required number of alternations then $r = (p^* - \eta p)/(q^* - \eta q)$ as defined in [1] can be shown to be a better approximation for sufficiently small $| \eta |$. In addition, if $r$ is not reduced, then it also can be shown that for sufficiently small $|\eta|$ it is possible to reduce $r$ to lowest terms so that the resulting rational function has a positive denominator on $X$ and hence belongs to $R_n^m(X)$. This proof can also be used to establish uniqueness.

If one drops the requirement that $(p, q) = 1$ and $q > 0$ on $X$ then Professor C. B. Dunham has informed us that this alternation behavior is no longer a necessary condition. Also, we would like to thank Professor Dunham for pointing out the reducing difficulty to us.

Likewise, one can prove a strong uniqueness result for $R_n^0(X)$. The proof is similar to the usual proof by contradiction for an interval. However, it is necessary to refer to the existence argument to guarantee that a particular $c/q(x)$ is such that $c > 0$ at one point of the argument. A copy of this proof is available upon request.

THEOREM 3. *Let $f \in C(X)$ satisfy $f(x) > 0$ for all $x \in X$. Then there exists a unique best approximation $1/p^*$ to $1/f$ from $R_n^0(X)$. Furthermore, there exists a positive constant $\gamma = \gamma(f)$ such that for each $1/p \in R_n^0(X)$,*

$$\left\| \frac{1}{f} - \frac{1}{p} \right\| \geqslant \left\| \frac{1}{f} - \frac{1}{p^*} \right\| + \gamma \left\| \frac{1}{p} - \frac{1}{p^*} \right\|$$

(*strong uniqueness*).

## 3. COMPUTATION

In this section, we wish to describe three possible algorithms for computing best approximations to $1/f, f(x) > 0$ for all $x \in X$, from $R_n^0(X)$. The algorithms are Remes, differential correction, and a hybrid algorithm which is a combination of the first two. In a future paper we shall report on numerical experiments involving these algorithms. In what follows, we shall assume $X$ is a finite set.

The Remes algorithm has been widely studied and appears often in the literature. Two explicit papers where the Remes algorithm is proposed for calculating best rational approximations are Cody, Fraser, and Hart [8], and Ralston [16]. The Remes algorithm consists of two main operations:

(i) the solution of a nonlinear system, and

(ii) the exchange of a certain set of points.

It is known that in general the nonlinear system may have many solutions (and sometimes none of which belong to $R_n^0(X)$) [10, 17, p. 104]. Thus Remes could fail to run due to its inability to either solve this system or by returning a solution to this system which is not in the class $R_n^0(X)$. (A second problem with the Remes algorithm will be mentioned later.) Even if the algorithm is able to solve this system at each step with a solution in $R_n^0(X)$, convergence can be guaranteed only if the algorithm was initialized with a sufficiently "good" starting approximation (for the case $X$ is an interval). In a recent study by Lee and Roberts [13], it is observed that the Remes algorithm is very fast when it converges, but may fail to converge. Also, observe that in the theory we are considering here every function being approximated is normal in the usual sense.

The differential correction algorithm was originally introduced by Cheney and Loeb [6]. This algorithm was shown to have very desirable convergence properties by Barrodale, Powell, and Roberts in [2] and a Fortran listing of it can be found in [12]. In the setting considered here this particular algorithm possesses guaranteed (quadratic) convergence to the desired best approximation in $R_n^0(X)$. However, in practice, if $X$ is large it is sometimes necessary to solve this problem on a subset of $X$ and then initialize the full problem with this solution. This is due to the fact that this method involves a linear programming subroutine which is sometimes numerically difficult to solve without a good initialization. Also, due to the inclusion of this linear programming subroutine, this algorithm is quite slow (some 19 times slower than Remes (when Remes converges) in tests done in [13]).

The final algorithm that we wish to mention for this problem is a hybrid of the above two. Precisely, we propose to replace the step of the Remes algorithm where a nonlinear system is solved to get a best approximation on a reference set (smaller than $X$) with the differential correction algorithm applied to this reference set to give the desired best approximation on this set. This method will eliminate the problems of the Remes algorithm associated with the solution of the nonlinear system in that a best approximation on the reference set (which is *positive* on the reference set) will be found. However, there is still no guarantee that the best approximation on a given reference set found by the differential correction algorithm will actually belong to $R_n^0(X)$ (i.e., it may fail to be positive (or defined) on some points of $X$ not in the reference set). In fact, we have encountered such examples in testing our hybrid algorithm and these examples have given rise to a second problem in the Remes algorithm. Namely, the exchange procedure cycled.

Thus, it is also necessary to modify the exchange procedure. At present we are testing two modified algorithms. In order to describe these modifications, let us assume that we are at the $k$th step of the iteration and suppose that the best approximation $r_k = 1/p_k$ on the reference set $X_k$ ($n + 2$ points from $X$) has been found by the differential correction algorithm.

The first modified exchange algorithm is as follows. Perform a multiple exchange in the usual manner only among those points of $X$ where $r_k > 0$ holds. If $r_k$ is not the best approximation on the set of points where it is positive then a new reference set is obtained and the algorithm proceeds to the differential correction phase to find the best approximation on this new reference set. If $r_k$ is the best approximation on the set of points where it is positive then terminate the algorithm if $r_k(x) > 0$ for all $x \in X$ ($r_k$ is the desired best approximation on $X$) or adjoin to the set $X_k$, $y \in X$ where $p_k(y) = \min\{p_k(x): x \in X\}$. Note that $p_k(y) \leqslant 0$ must hold in this case. Set $X'_{k+1} = \{y\} \cup X_k$ and apply the differential correction algorithm to this set of $n + 3$ points finding the best approximation $r_{k+1}$ on it. Next, reduce $X'_{k+1}$ to a subset of $n + 2$ points, $X_{k+1}$, where $X_{k+1}$ is chosen so that alternation holds on $X_{k+1}$. Now repeat the exchange procedure on $X_{k+1}$ with respect to $r_{k+1}$.

The second modified exchange algorithm is basically a reordering of the above one. In particular, if $r_k$ the best approximation on $X_k$ is positive on all of $X$ then we proceed with a multiple exchange in the usual manner. If $r_k$ is not positive on all of $X$ then we adjoin $y \in X$ to $X_k$ precisely as above, getting $X'_{k+1}$ and proceed as in the above algorithm.

One can prove that in both of these modified algorithms, cycling cannot occur and that global convergence holds for $X$ finite (i.e., error of approximation on successive reference sets strictly increases). In a future paper we shall give a detailed description of these two algorithms and report on the results of our numerical testing of them. Also, we are studying the extension of these ideas to $R_n^m(X)$ and will also report on this at that time.

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